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LETTER TO THE EDITOR

Unbinding transition in a many-string system

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Abstract. A $1+1$ dimensional model of n non-intersecting strings with short-range attractive interactions on a lattice is solved exactly. For arbitrary finite $n > 2$ there is a second-order binding–unbinding transition with the same critical exponents as for $n = 2$. In the limit $n \to \infty$ the transition becomes first order.

We consider a system of n thermally-fluctuating strings on a square lattice, as shown in figure 1. The strings are non-intersecting and subject to the solid-on-solid restriction, meaning that each string crosses a horizontal line across the system only once. The strings are under tension, i.e. changes in the coordinate x cost energy, and they exert short-range forces on each other. Systems similar to this are of interest in connection with the wetting transition [1], the commensurate–incommensurate transition [2], the unbinding transition in membranes [3], and the statistics of 'drunken walkers' [4]. In this letter we will be mainly interested in the binding–unbinding transition that takes place in the presence of short-range attractive forces between strings as the temperature or the interaction strength is varied.

The statistical weight of the n-string system is conveniently specified (see figure 1) in terms of the transfer matrix $\langle x_1', \ldots, x_n' | T(y'-y) | x_1, \ldots, x_n \rangle$. The restriction that at most one string passes through a site is reminiscent of fermion statistics, and we use the second-quantized representation $c_{x_1}^{+} \ldots c_{x_n}^{+} |0\rangle$ for the state vector $|x_1 \ldots x_n\rangle$. Here c_x and c_x^{+} are conventional annihilation and creation operators for spinless fermions, and $|0\rangle$ denotes the fermion vacuum state.

In this paper we consider the model with transfer operator

$$T(y'-y) = \exp[-H(y'-y)] \tag{1a}$$

$$H = -R \sum_{i=1}^{N_x} (c_{i+1}^{+} c_i + c_i^{+} c_{i+1} + U c_{i+1}^{+} c_{i+1} c_i^{+} c_i). \tag{1b}$$

In equation (1) the index i rather than x is used to avoid confusion with the xxz spin model to be encountered below. Periodic boundary conditions, i.e. $c_1 = c_{N_x+1}$, are imposed.

Equation (1) defines the transfer matrix for arbitrary continuous $y' - y$, not just for integer values. Expanding $T(y'-y)$ to first order in $y'-y$, we see that the quantity R, which is taken to be positive, is a monotonically decreasing function of the string tension. The quantity RU is positive for an attractive force between adjacent strings with a range of one lattice constant and negative for a repulsive force. The parameter R can be eliminated

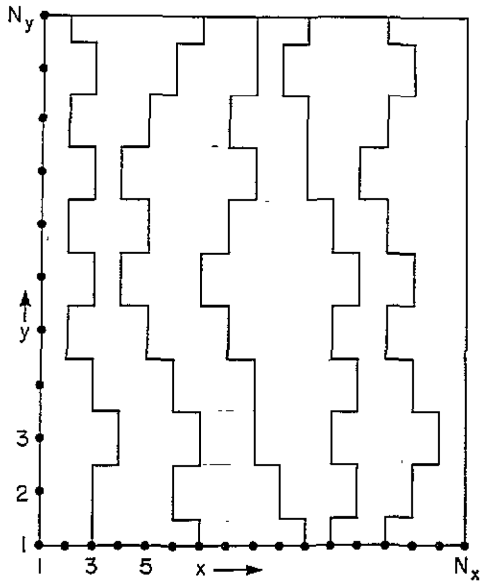

Figure 1. System of five strings.

from equation (1) by the rescaling $y \to y/R$ of lengths in the y direction, and with no loss of generality we set $R = 1$ from now on.

In earlier work [5] we gave an exact analysis of the model (1) for an *infinite* number of strings occupying a finite fraction ρ of the lattice sites. We considered the double thermodynamic limit $n \to \infty$, $N_x \to \infty$ with $0 < \rho < 1$, where

$$\rho = \lim_{\substack{n\to\infty \\ N_x\to\infty}} \frac{n}{N_x}. \tag{2}$$

Some of our results for strings were first derived in the context of the quantum lattice gas by Yang and Yang [6].

In this paper we study the binding–unbinding transition of *finite* numbers of strings on an infinite lattice. The limit $N_x \to \infty$ is taken holding n fixed. The case $n = 3$ is of particular interest. Both first-order [7] and second-order [8] unbinding transitions have been reported for three-string models with the same general features as the one we consider.

With the Wigner–Jordan transformation $S_i^+ = c_i^+ \exp(\mathrm{i}\pi \sum_{j<i} c_j^+ c_j)$, the fermion Hamiltonian (1), with $R = 1$, may be rewritten [9] as $H = H^{xxz}$, where

$$H^{xxz} = -2\sum_i [S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \tfrac{1}{2}U(S_i^z S_{i+1}^z - \tfrac{1}{4}) + \tfrac{1}{2}U(S_i^z + \tfrac{1}{2})] \tag{3}$$

is the quantum spin-1/2 xxz Heisenberg Hamiltonian.

The magnetization, which commutes with H^{xxz}, and the number of strings n are related by

$$n = \sum_{i=1}^{N_x} (S_i^z + \tfrac{1}{2}). \tag{4}$$

According to equations (1) and (3), in the limit $N_y \to \infty$ the minimum energy $E_0(n)$ of the xxz chain for a given n represents the free energy of the system of n strings.

The xxz model is soluble by the Bethe ansatz and has been studied extensively [6, 10, 11]. In the limit $N_x \to \infty$ with finite $n \geqslant 2$

$$E_0(n) = \begin{cases} -2n & U < 2 \\ -2\sum_{\alpha=1}^{n} \cosh \kappa_\alpha & U > 2 \end{cases} \tag{5a}$$

where

$$U e^{\kappa_\alpha} - 1 - e^{\kappa_\alpha + \kappa_{\alpha+1}} = 0 \tag{5b}$$

$$\sum_{\alpha=1}^{n} \kappa_\alpha = 0 \tag{5c}$$

and the κ_α are ordered so that $\kappa_{\alpha+1} < \kappa_\alpha$. For $U > 2$ the state of minimum energy is a bound state of the n fermions. The wave numbers $p_\alpha = \mathrm{i}\kappa_\alpha$ are imaginary, i.e. the κ_α are real, reflecting the real exponential character of the Bethe ansatz wave function. For $U < 2$ the fermions are unbound in the state of minimum energy. The corresponding wave numbers are real and vanish in the limit $N_x \to \infty$. For all $n \geqslant 2$ the unbinding transition takes place at $U = 2$.

With the substitutions

$$U = 2\cosh\phi \qquad \phi > 0 \tag{6a}$$

$$\tanh\frac{\kappa_\alpha}{2} = \tanh\frac{\phi}{2}\tanh\frac{\psi_\alpha}{2} \tag{6b}$$

the recurrence relation (5b) takes the simpler form

$$\tanh[\tfrac{1}{2}(\psi_\alpha - \psi_{\alpha+1})] - \tanh\phi = 0. \tag{7}$$

Thus $\psi_{\alpha+1} = \psi_\alpha - 2\phi$, which with (5$c$) and (6$b$) implies

$$\tanh\frac{\kappa_\alpha}{2} = \tanh\frac{\phi}{2}\tanh\left[\left(\frac{n+1}{2} - \alpha\right)\phi\right]. \tag{8}$$

The critical behaviour as U approaches 2 may be studied by expanding equations (5a) and (8) in powers of $U - 2$. This yields

$$\kappa_\alpha = \left(\frac{n+1}{2} - \alpha\right)(U-2) + \mathrm{O}[(U-2)^2] \tag{9a}$$

$$E_0(n) = -2n - \tfrac{1}{12}n(n^2-1)(U-2)^2 + \mathrm{O}[(U-2)^4] \tag{9b}$$

for $U > 2$.

Comparing equations (5a) and (9b), one sees that the free energy $E_0(n)$ and its first derivative with respect to U are continuous at $U = 2$, whereas the second derivative is discontinuous. Thus the unbinding transition of the strings is second order, with a discontinuity in the specific heat $C \propto \partial^2 E_0(n)/\partial U^2$.

As in [8] we define critical exponents ψ, $\nu_\perp$, $\nu_\parallel$ by

$$C \sim (U - U_c)^{-\alpha} \qquad \langle \ell \rangle \sim (U - U_c)^{-\psi} \tag{10a}$$

$$\xi_\perp \sim (U - U_c)^{-\nu_\perp} \qquad \xi_\parallel \sim (U - U_c)^{-\nu_\parallel} \tag{10b}$$

as U approaches the critical value $U_c = 2$ from above. Here $\langle \ell \rangle$ denotes the average separation of the strings, and $\xi_\perp$ and $\xi_\parallel$ the correlation lengths with which the correlation function

$$G(x', y'; x, y) = \langle E_0(n)|c_{x'}^{+}c_{x'} \exp(-[H - E_0(n)](y' - y))c_x^{+}c_x|E_0(n)\rangle \tag{11}$$

decays in the x and y directions, respectively.

The quantities $\langle \ell \rangle$ and $\xi_\perp$ have the same critical behaviour as the attenuation lengths κ_α^{-1} in the Bethe ansatz wave function. Equations (9) and (10) imply

$$\alpha = 0 \qquad \psi = \nu_\perp = 1. \tag{12a}$$

From equations (1), (3) and (11) one sees that $\xi_\parallel^{-1} = E_1(n) - E_0(n)$, where $E_1(n)$ is the energy of the lowest excited state for which $\langle E_1(n)|c_x^{+}c_x|E_0(n)\rangle \neq 0$. Presumably this state corresponds to $n - 1$ bound strings and one free string with zero momentum. Thus from equations (5a) and (9)

$$\nu_\parallel = 2. \tag{12b}$$

The exponents in equation (12) are independent of n. For arbitrary finite $n > 2$ the unbinding transition is in the same universality class as for $n = 2$. This transition, in turn, is in the same universality class as the much investigated depinning transition of a 1+1 dimensional string attracted to the boundary of a semi-infinite system by a short-range force [1].

Netz and Lipowsky [8] have carried out numerical studies of the unbinding of three strings interacting in 1+1 dimensions with hard-core square-well potentials. The fluctuations of the interior string generate a repulsive effective potential between the two outer strings proportional to the inverse square separation. This particular spatial dependence leads to non-universal critical exponents in two-string systems [7, 12], and Netz and Lipowsky find that the critical exponents ψ, $\nu_\perp$, $\nu_\parallel$ of three strings decrease as the tension of the interior string is lowered with the other two tensions held fixed. There is no indication of non-universality in the exact critical exponents (12) for the model defined by (1). However, in the Hamiltonian (1) all the strings have the same tension.

We now turn to the limit $n \to \infty$, having already taken the limit $N_x \to \infty$ in obtaining equations (5)–(9). The coefficient of $(U - 2)^2$ in equation (9) diverges as n^3 for large n, signalling a change in the nature of the transition. We shall see that the transition becomes first order.

In the limit $n \to \infty$ the sum in equations (5a) and (8) can be evaluated analytically [6] for arbitrary $U > 2$, with the result

$$\sum_{\alpha=1}^{\infty}(U - 2\cosh\kappa_\alpha) = (U^2 - 4)^{1/2}. \tag{13}$$

Together with equation (5a) this implies

$$e(U) = \lim_{n\to\infty} \frac{E_0(n)}{n} = \begin{cases} -2 & U < 2 \\ -U & U > 2\,. \end{cases} \tag{14}$$

The quantity $e(U)$ is the free energy per string. Its derivative with respect to U has the discontinuity

$$\frac{\partial e(U=2-)}{\partial U} - \frac{\partial e(U=2+)}{\partial U} = 1 \tag{15}$$

corresponding to the latent heat per string in the unbinding transition. Thus the transition is first order.

As mentioned above, in earlier work [5] we studied system (1) in the limit (2) of an infinite number of strings with a finite density ρ. For $0 < \rho < 1$ there is a first-order transition at $U = 2$, with a clustered or bound string phase for $U > 2$. For $U < -2$ and $\rho = 1/2$ (half filling of the lattice), the system exhibits long-range staggered order.

Presumably if the limit $n \to \infty$ is taken *after* the limit $N_x \to \infty$, as in deriving the free energy per string $e(U)$ given in equation (14), the number of strings, though infinite, always remains negligible in comparison with the system size, i.e. $\rho = 0$. We now show that the free energy per string $\tilde{e}(U, \rho)$, calculated in the double thermodynamic limit (2), does indeed reduce to $e(U)$ in the limit $\rho \to 0$.

Defining $\tilde{e}(U, \rho)$ to be the lowest eigenvalue of H^{xxz}/n for fixed ρ in the limit $n, N_x \to \infty$, we note the relation

$$\rho\tilde{e}(U, \rho) = 2f(U/2, 2\rho - 1) + \tfrac{1}{4}U - U\rho \tag{16}$$

with the function $f(\Delta, y)$ analysed in detail by Yang and Yang [10, 11]. For $U > 2$, $f(\Delta, y) = -\Delta/4$, corresponding to a ferromagnetic ground state. Substitution of this result and the asymptotic form (see equation (70) in [11]) of $f(\Delta, y)$ for $y \to -1$, $\Delta < 1$ yields

$$\tilde{e}(U, \rho) = \begin{cases} -2 + \dfrac{\pi^2}{3}\rho^2\left(1 - \dfrac{2U}{2-U}\rho\right) + \mathrm{O}(\rho^4) & U < 2 \\ -U & U > 2 \end{cases} \tag{17}$$

where the $\mathrm{O}(\rho^4)$ correction also depends on U. Thus $\tilde{e}(U, \rho)$ does indeed reduce to $e(U)$ given in (14) in the limit $\rho \to 0$.

In concluding we briefly summarize the phase diagram of the system of strings in the variables U, ρ and the nature of the corresponding phase transitions. More details can be found in [5, 6, 10, 11, 13].

The free energy per string $\tilde{e}(U, \rho)$ is analytic in U and ρ except on the line $U = 2$ and the half line $U < 2$, $\rho = 1/2$. For $U > 2$ the strings are bound, and the pressure

$$P = \rho^2 \frac{\partial \tilde{e}}{\partial \rho} \tag{18}$$

vanishes identically, as follows from (17). The transition at $U = 2$ is first order, i.e. $\partial\tilde{e}/\partial U$ is discontinuous. This can be seen in the limit $\rho \to 0$ from equation (17) and for $\rho = 1/2$ from equation (45) in [11].

The special role of $\rho = 1/2$ (half filling of the lattice) is related to particle-hole symmetry in the fermion model (1) or up-down symmetry [10, 11] in the xxz model (3), which imply $f(\Delta, y) = f(\Delta, -y)$ in equation (16). For $U < -2$, $\rho = 1/2$ the strings have long-range staggered order. There is 'Luttinger liquid' behaviour [13] for $U < 2$ except on the line $\rho = 1/2$.

For $U < -2$ the pressure P given by (18) is discontinuous at $\rho = 1/2$. For $-2 < U < 2$, P is continuous at $\rho = 1/2$, but $\partial^m \tilde{e}/\partial\rho^m$ diverges for $m > 2+4\mu/(\pi-\mu)$, where $U = -2\cos\mu$, $0 < \mu < \pi$. At $U = -2$, $\rho = 1/2$ the free energy $\tilde{e}(U, \rho)$ has an essential singularity. All its derivatives with respect to U are continuous at this point.

Finally we recall that in the Luttinger-liquid phase the correlations of the strings decay as power laws with non-universal U-dependent exponents [5, 13]. This is quite a different manifestation of non-universality from the non-universality in the unbinding of three strings reported in [8].

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